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Correction to: Approximations and Generalized Newton Methods [Math. Program., Ser. B (2018) 168: 673–716]

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Abstract We point out two erroneous statements in the paper [D. Klatte, B. Kummer, Approximations and Generalized Newton Methods, Math. Program., Ser. B (2018) 168: 673–716, DOI: https://doi.org/10.1007/s10107-017-1194-8] concerning the directional differentiability of functions which allow an approximation by certain Newton maps. We thank Helmut Gfrerer who discovered these errors and created a counterexample which will be presented, too.

Keywords Newton map · semismoothness · Clarke's Jacobian · directional differentiability

Mathematics Subject Classification (2010) 49J53 · 49K40 · 90C31 · 65J05

In Subsection 3.2 of the paper [5], we studied the interrelations between semismoothness (in the sense of Qi, Sun [7]), approximation by Newton maps (in the sense of [4,6]) and directional differentiability. In a private communication [3], Helmut Gfrerer pointed out that Theorem 6 in [5] is incorrect, and he created a counterexample showing that Proposition 2 in [5] is false.

First we present the correct statement of [5, Theorem 6]. Suppose that f is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (briefly $f \in C^{0,1}(\mathbb{R}^n, \mathbb{R}^m)$). Let $f'(\bar{x}; u)$ be the standard directional derivative of f at $\bar{x}$ in direction u, and denote by $\partial^{CL} f(\bar{x})$ Clarke's generalized Jacobian [1,2] of f at $\bar{x}$. For the definitions of Newton maps and semismoothness, we refer to [5].

Theorem ([5, Theorem 6] corrected). f is semismooth at $\bar{x}$ if and only if
(i) $\partial^{CL} f$ is a Newton map for f at $\bar{x}$, and
(ii) $f'(\bar{x}; u)$ exists for each u.

Indeed, by [7, Prop. 2.1 & Thm. 2.3], semismoothness of f at $\bar{x}$ implies both (i) and (ii). On the other hand, the if-direction of the theorem follows from [7, Thm. 2.3], by taking $f'(\bar{x}; u) = f(\bar{x} + u) - f(\bar{x}) + o(u)$ under (ii) into account (for the latter see, e.g., [4, Lemma A2], [8]).

The original version of [5, Theorem 6] included the incorrect statement that (ii) follows from (i), while [5, Proposition 2] claimed that even a Newton-map property weaker than (i) implies (ii). Note that in the proof of Proposition 2 in [5], the estimate $[\ldots \geq 2$ if $h_k \geq 0]$ on line 6 of page 690 is false.

We finish this corrigendum by Gfrerer's counterexample [3]: it gives a function $f \in C^{0,1}(\mathbb{R}, \mathbb{R})$ which satisfies the Newton-map property (i), but is not directionally differentiable and hence not semismooth.

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Example. Consider two real sequences $a_k \downarrow 0$, $b_k \downarrow 0$ given by

$$a_1 := 1,\ b_k := e^{-2k} a_k,\ a_{k+1} := e^{-2k} b_k,\ k \geq 1,$$

and the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & \text{if } x \geq 1 \\ -x + \frac{1}{k} x \ln \frac{x}{b_k} & \text{if } x \in [b_k, a_k),\ k \geq 1, \\ x - \frac{1}{k} x \ln \frac{x}{a_{k+1}} & \text{if } x \in [a_{k+1}, b_k),\ k \geq 1, \\ 0 & \text{if } x \leq 0. \end{cases}$$

f is continuous, since for $k \geq 1$,

$$\begin{aligned} \lim_{x \uparrow a_k} f(x) &= \lim_{x \uparrow a_k} (-x + \tfrac{1}{k} x \ln \tfrac{x}{b_k}) = -a_k + \tfrac{1}{k} a_k \ln \tfrac{a_k}{b_k} = -a_k + \tfrac{1}{k} a_k 2k = a_k = f(a_k), \\ \lim_{x \uparrow b_k} f(x) &= \lim_{x \uparrow b_k} (x - \tfrac{1}{k} x \ln \tfrac{x}{a_{k+1}}) = b_k - \tfrac{1}{k} b_k \ln \tfrac{b_k}{a_{k+1}} = b_k - \tfrac{1}{k} b_k 2k = -b_k = f(b_k), \end{aligned}$$

while $\lim_{x \downarrow 0} f(x) = 0 = f(0)$ because of

$$\begin{aligned} -x \leq f(x) &\leq -x + \tfrac{1}{k} x \ln \tfrac{a_k}{b_k} = -x + \tfrac{1}{k} x 2k = x,\ x \in [b_k, a_k), \\ -x = \ x - \tfrac{1}{k} x 2k &= x - \tfrac{1}{k} x \ln \tfrac{b_k}{a_{k+1}} \leq f(x) \ \leq x,\ x \in [a_{k+1}, b_k). \end{aligned}$$

f is continuously differentiable except for the points 0, a_k, b_k $(k \geq 1)$, with derivative

$$f'(x) = \begin{cases} 1 & \text{if } x > 1, \\ \frac{f(x)}{x} + \frac{1}{k} & \text{if } x \in (b_k, a_k), \\ \frac{f(x)}{x} - \frac{1}{k} & \text{if } x \in (a_{k+1}, b_k), \\ 0 & \text{if } x < 0. \end{cases}$$

Then we have

$$\partial^{CL} f(x) = \begin{cases} [\,1, 2\,] & \text{if } x = a_1, \\ \left[\,1 - \frac{1}{k}, 1 + \frac{1}{k+1}\,\right] & \text{if } x = a_{k+1},\ k \geq 1, \\ [\,-1 - \frac{1}{k}, -1 + \frac{1}{k}\,] & \text{if } x = b_k,\ k \geq 1, \\ [\,-1, 1\,] & \text{if } x = 0, \\ f'(x) \in [\,-2, 2\,] & \text{else} \end{cases}$$

and Lipschitz continuity of f. Thus, for any $x < 1$ and any $A \in \partial^{CL} f(x)$, one has

$$|f(x) - f(0) - A(x - 0)| \begin{cases} = \frac{x}{k} & \text{if } x \in (a_{k+1}, a_k) \setminus \{b_k\},\ k \geq 1, \\ \leq \frac{x}{k} & \text{if } x = b_k \text{ or } x = a_{k+1},\ k \geq 1, \\ = 0 & \text{if } x \leq 0. \end{cases}$$

Hence, by definition, $\partial^{CL} f$ is a Newton map for f at $\bar{x} = 0$. On the other hand,

$$\lim_{k \to \infty} \frac{f(a_k) - f(0)}{a_k - 0} = 1, \quad \lim_{k \to \infty} \frac{f(b_k) - f(0)}{b_k - 0} = -1,$$

and so $f'(0; 1)$ does not exist. Consequently, f is not semismooth at $\bar{x} = 0$.

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